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ON A CIRCULAR CORRELATION TECHNIQUE FOR FILTERING
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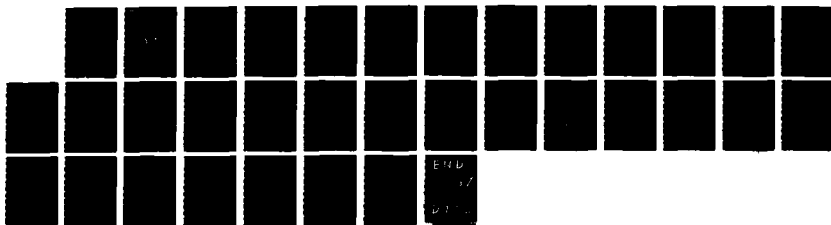
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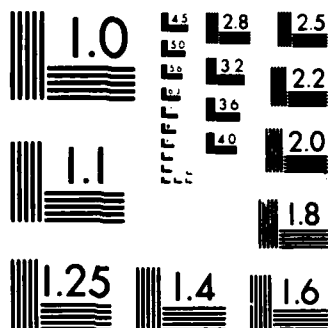
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ON A CIRCULAR CORRELATION TECHNIQUE FOR
FILTERING RANDOMLY NOISY DATA

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ABSTRACT

This paper introduces a technique of smoothing high frequency random fluctuations from noisy data. An optimum lowpass filter is developed by using the circular serial correlation statistic to minimize the random noise content in serial data. The procedure is applicable for statistically mild data. All the basic aspects of the new filter technique are discussed and are applied to both simulated noisy data and actual measured data of unknown noise characteristics. Not only is the procedure applicable to noise reduction, but it also appears to provide a means to reconstruct undersampled data.

Introduction

There are several methods of estimating the intelligence, or signal, components of noisy data [1,2]. Most of these methods involve a least-square error approach either directly or indirectly [3]. In the conventional least square (LS) approaches, an error signal is used to determine a set of system parameters, usually the constant coefficients in a difference equation. In some of these LS approaches, either the order of the data is required or its estimate, since the number of coefficients is often assumed to be known. Whenever the system order is unbounded, as occurs in the singularity expansion method, it is expected that the conventional LS procedures will have difficulties [4].

It is well known that a truncated Fourier series provides an LS fit to any function that is bounded and piecewise differentiable in a finite interval [5]. A procedure is presented for obtaining the optimum truncation by



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examining the randomness of the time series formed by the differences between the original data and the truncated Fourier series. The relative randomness is determined by using the circular serial correlation coefficient [6]. Since the truncated Fourier series is equivalent to the result of implementing a low pass filter, the foregoing procedure yields the optimum lowpass filter in reducing the random noise contribution to the data, provided the data are statistically mild.

In other methods, for example, adaptive [7-9] and Kalman [10,11] filtering, a prior knowledge of noise components mean, variance and correlation properties are needed. In adaptive filtering it is well known that the best LS estimate of the primary signal is produced by adjusting filter parameters, i.e., minimizing the LS error power. At the expense of not necessarily being the best LS representative, our adaptive LS estimate is produced by maximizing the randomness of the error through changing the set of Fourier coefficients. Also, it is possible to recast the Fourier series in terms of state variables for Kalman filtering analysis. The solution to this problem, however, additionally assumes a known initial state and involves a set of matrix inversions. Our nonparametric, orthogonal polynomial approach is a simple straight forward method, only requiring a statistical test of randomness of the error signal as opposed to using ad hoc rules of thumb in making this type test. The test statistic only requires $2N$ additions and multiplications calculations. The statistic used in making the randomness test is similar to the non-normalized version of the correlation statistic recently used by Pakula and Kay [12] in their radar detection work. It is a circularly defined single lag autocovariance statistic well known in the

mathematical statistics area[13,14] and the spectral analysis detection area[15,16,17]. However, in our application of the "circular correlation" statistic, a bow-tie correction[17,18] for spectral bias error and windowing, is not required since the correlation function is calculated directly in the "time" domain. The method presented here is a simple utilization of a novel randomness test to construct a low-pass filter.

The flow diagram for the circular serial autocorrelation filter (serial correlator filter) is shown in Fig. 1. It shows the principle processing used to obtain an estimate signal from the noisy signal. The basic aspects of this new technique for filtering noisy data are considered in the following discussion. In the first section, fundamental characteristics of a Fourier integrable noisy signal are presented. The next section includes a discussion of the estimation of the desired signal. Also presented in this section are the statistical bases of the filter which allow the extraction of an estimate of the desired signal. Salient features of the correlation coefficient and what is meant by statistically mild data are also discussed. The third section provides a brief treatment of the noise estimation problem. In this section, rough estimates of signal-to-noise ratios related to the noisy signal and to the estimated desired signal are presented. Lastly, several examples of the actual use of the filter are given.

Noisy Signals

A real noisy signal is assumed to have two summable components: a deterministic component and a non-deterministic component. The deterministic component is called intelligence and the non-deterministic component is called

FLOW DIAGRAM

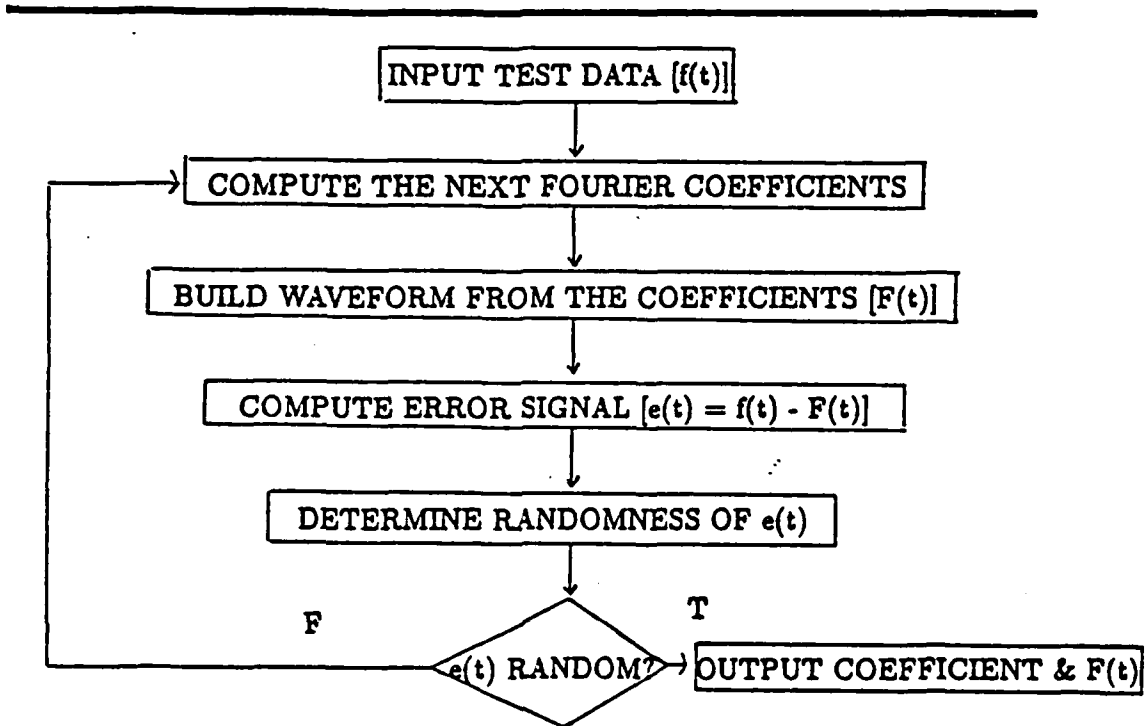


Figure 1: Basic Diagram of Correlation Filter

original noise. If f_n , f , and n respectively denote the noisy signal, intelligence signal, and original noise signal, then f_n obeys the relation

$$f_n(t) = f(t) + n(t) \quad (1)$$

where t is the independent variable. $n(t)$ for each t is a value assumable by a stationary random noise variable. $f(t)$ is assumed to be a response of a deterministic and stationary system. In discrete cases for N even samples of sampling period T , where subscript k denotes the discrete independent variable, the noise contaminated signal of equation (1) is represented by

$$f_{nk} = f_k + n_k \quad k=0,1,\dots,N-1. \quad (2)$$

The intelligence signal-to-original noise ratio in dB, defined as the ratio of the rms value of f_k to the rms value of n_k , is given by the usually unknown quantity*

$$SNR_1 = 20 \log \frac{\text{rms}(f_k)}{\text{rms}(n_k)} \quad (3)$$

To determine an estimate** of SNR_1 and to eliminate portions of n_k , it is sufficient to determine the estimates of the intelligence signal f_k (the

*The rms value of a discrete signal is approximately its standard deviation, if its average value is negligible.

**An estimated quantity is denoted by a small circumflex mark over the top of the basic symbol. If the value of the subscript is not indicated, the subscripted quantity represents the entire waveform.

symbol \hat{f}_k) and n_k (the symbol \hat{n}_k). These estimates are discussed in the following sections.

Estimation of Intelligence

In some conventional LS approaches either the order of the data is required or its estimate. The truncated Fourier series estimate of the noise contaminated signal, however, eliminates the necessity of prior knowledge of order. On the other hand, the Fourier series (\hat{f}_{nk}) representation retains portions of random noise components which share the same frequencies as the harmonics in the representation. This distorts the series representation. Since in finite applications the Fourier series must truncate at some harmonic (L), the difference signal f_{nk} minus \hat{f}_{nk} may contain deterministic as well as non-deterministic components. This causes additional concerns. These two problems are related since the Fourier series is the noisy signal minus the difference. In the special case of a zero mean n_k , it is easily shown that the estimated signal (\hat{f}_{nk}) is exact for a diagonal autocovariance difference matrix; thus, the latter problem is eliminated because in this case the difference signal is purely random.

As shown in Appendix A a stationary random error (the difference between intelligence and its estimate) implies an intelligence estimate whose deterministic part is the ideal intelligence f_k . It also is easily shown that when intelligence is estimated exactly, the error is random regardless of the noise mean. We assume when the difference signal is significantly random that \hat{f}_{nk} is a reasonable estimate of f_k . That is, the estimate of the intelligence is related to the Fourier series estimate for significantly random error by the approximation

$$\hat{f}_k|_L = \hat{f}_{nk} \quad , \quad (4)$$

where L signifies the dependency of the estimate of f_k on the number of harmonics retained in the Fourier series. By definition, therefore, we take as an estimate of the intelligence, the following truncated Fourier series, where $\omega_0 = 2\pi/[(N-1)T]$ is the fundamental frequency:

$$\hat{f}_k|_L = A_0 + \sum_{m=1}^L [A_m \cos(m\omega_0 kT) + B_m \sin(m\omega_0 kT)]. \quad (5)$$

The above Fourier coefficients are given by either a fast Fourier transform of a properly sampled* f_{nk} or by some archaic evaluation of the following integrals over a fundamental period:

$$A_0 = \frac{1}{T_0} \int_0^{T_0} f_n(t) dt \quad (6)$$

$$A_m = \frac{2}{T_0} \int_0^{T_0} f_n(t) \cos(m\omega_0 t) dt \quad m=1, \dots, L \quad (7)$$

$$B_m = \frac{2}{T_0} \int_0^{T_0} f_n(t) \sin(m\omega_0 t) dt \quad m=1, \dots, L \quad (8)$$

* If data are not properly sampled aliasing causes the DFT method to give poor results for the coefficients.

$f_n(t)$ is obtained from the sample f_{nk} via interpolation. Both linear and quadratic interpolation have been considered with comparable results obtained in all cases.

In attempting to generate a non-deterministic curve-fit error the order, L , of the harmonic expansion is incremented to its maximum value determined by the Nyquist sampling interval. For each L an error signal is defined by the difference between the noisy signal and the intelligence estimate:

$$e_{Lk} = f_{nk} - \hat{f}_k|_L \quad . \quad (9)$$

This signal has certain characteristics. By definition, it tends not to be slowly oscillatory. It is not the difference between \hat{f}_k and the unknown f_k . Its rms value, a figure of merit denoted by E_L ,

$$E_L = \text{rms} (e_{Lk}) \quad , \quad (10)$$

is slowly varying over a range of L values. Thus, its minimum power as a function of L is not necessarily located at a finite turning point. Finally, its randomness is a function of L . A method of deciding when this signal is random follows.

Another figure of merit of the error signal is its circular serial correlation coefficient. An exact distribution of the (circular) serial correlation coefficient for random time series is known [6,13,14]. The Wald and Wolfowitz "correlation" statistic [6] for N sample values of detrended e_{Lk} is

$$R_L = e_{L,0}e_{L,N-1} + \sum_{k=1}^{N-1} e_{L,k-1}e_{L,k} \quad (11)$$

This simple single lag correlation measure is a function of L . R_L has a mean and variance for random signals given respectively by

$$\mu_{R_L} = (S_1^2 - S_2)/(N-1) \quad (12)$$

$$\sigma_{R_L}^2 = \frac{S_2^2 - S_4}{N-1} + \frac{S_1^4 - 4S_1^2S_2 + 4S_1S_3 + S_2^2 - 2S_4}{(N-1)(N-2)} - \frac{(S_1^2 - S_2)^2}{(N-1)^2} \quad (13)$$

where S_i is the i^{th} moment of the detrended e_{Lk} about the origin. Other statistical characteristics of the error are also central to this work, and we need more than the above correlation moments to structure a randomness test.

At this point, statistically mild data is defined and relevant theorems cited. Statistically mild data are data 1) whose moments are bounded for all N , and 2) whose variance is greater than zero for $N \rightarrow \infty$. (N is greater than 75 for asymptotic properties of the data.) In applications of our method, if data values are bounded and no single value is exactly equal to the mean of the data, then the data are statistically mild as provided in Theorems 1 and 2 of Appendix B. Thus, each finite error value should be compared to the error mean to test for mildness before the above correlation statistics are applied.

The distribution of R_L is asymptotically normal for statistically mild data. For "large" N a test statistic which is a normalization of R_L ,

$$Z_L = \frac{R_L - \mu_{R_L}}{\sigma_{R_L}}, \quad (14)$$

can be used to structure a statistical test of the hypothesis that e_{Lk} is random [14]. The test for randomness proceeds as follows. The probability that $|Z_L|$ has a value less than or equal to a positive Z_0 is

$$P(|Z_L| \leq Z_0) = \sqrt{\frac{2}{\pi}} \int_0^{Z_0} \exp[-u^2/2] du. \quad (15)$$

The null hypothesis (H_0) is that the error signal is random, i.e., $Z_L=0$; and the single alternate hypothesis (H_A) is that e_{Lk} is not random (e_{Lk} is oscillatory or has a trend), i.e., $Z_L \neq 0$. Utilizing the .05 significance level, the null hypothesis is rejected if $|Z_L| > 1.96$. Conversely, the null hypothesis is accepted if $|Z_L| < 1.96$. Consequently, in the search for an optimum L , the test statistic Z_L is calculated at each step and the value of L for which $|Z_L|$ is smallest is the optimum value provided $|Z_L| < 1.96$. However, it is important to note that there is not a 100% certainty the error is random even if a Z_L is found which is exactly zero. In this case, the outcome of the test for randomness would be exactly the same as it would be if the Z_L were 1.95. Therefore, we avoid the proposition that a $Z_L = 0$ implies statistically independent error samples; and more importantly, we use the confidence of the truth of the alternate hypothesis in deciding the randomness of the error samples.

There obviously are other approaches to determining an optimum L . The rms value of the error signal as described through equation (10) may be useful in this purpose. Its value can be minimized, as is done in many other

filtering methods. In cases of f_{nk} 's for which either the noise components are known to be small or Z_L is not minimizable in $[1, N/2)$, setting a limit on the curve-fit error can be an important requirement for determining the estimate waveform. Then iterations for selection of an optimal L can be stopped when the error is either less than the set limit or locally minimum. If large noise components are known to be present, however, the former statistical approach should be used. As shown in the first example, minimizing the error may not improve the errors randomness and may not be useful. In cases where the Fourier series is not the desired method to use in generating the error signal, other eigenfunction expansions can be used.

Generally, f_k can be represented by a weighted sum of orthonormal functions (ϕ_m 's) orthogonal over some determined interval, $T_a \leq t \leq T_b$, i.e.,

$$\hat{f}_k|_L = \sum_{m=0}^L C_m \phi_m(kT) \quad (16)$$

where the constants are given by discrete versions of the integrations

$$C_m = \left[\int_{T_a}^{T_b} f_n(t) \phi_m(t) dt \right] / \left[\int_{T_a}^{T_b} \phi_m^2(t) dt \right] \quad m=0, \dots, L. \quad (17)$$

When a more general set of expansion functions is used, the error signal and error figure of merits can be defined as in the Fourier LS approach. All of the preceding ideas can be restated in terms of the new expansion. In any signal estimation method used, however, the resulting random error signal is an estimate of the random noise content of f_{nk} .

Estimation of Noise

An estimate of n_k is taken to be the error signal when the number of terms in the series is optimum. It follows from the form of equation 3 that an estimate of the relative random noise content of f_{nk} is simply 20 times the common log of the ratio of the rms value of the estimates of f_k to the rms value of the estimate of n_k :

$$SNR_2 = 20 \log \frac{\text{rms}(\hat{f}_k|_{L'})}{\text{rms}(e_{L',k})} \quad (18)$$

where L' is the optimum value of L . Other estimates of SNR_1 are possible if various combinations of error signals and estimates of f_k are used in equation 18.

The noise content of the filtered signal also can be estimated. The SNR given by the following formula is an estimate of the SNR of \hat{f}_{nk} :

$$SNR_3 = 20 \log \frac{\text{rms}(\hat{f}_k)}{\text{rms}(f_{nk} - \hat{\hat{f}}_k - e_{L',k})} \quad (19)$$

where $\hat{\hat{f}}_k$ is an estimate of \hat{f}_k . SNR_3 is a measure of the remaining noise in the estimate of the intelligence signal. Equation 19 defines a good estimate when $\hat{\hat{f}}_k$ and \hat{f}_k are approximately equal. For large differences between $\hat{\hat{f}}_k$ and \hat{f}_k , equation 19 should not be used. The utility of the former equation 18 is indicated in the following discussion of application examples.

Application Examples

Plots of original noisy signal results are given in Figs. 2-6. Figure 2 represents simulated data which has a 20 dB SNR and its continuous parent intelligence obeys the following equation:

$$f(t) = 3e^{-5t} \sin 10t + e^{-t} \cos 5t. \quad (20)$$

Figures 3 and 4 are the error figures and Z-statistic plots respectively for the simulated data. Figure 3 shows that the error does not change much between 15 and 40 harmonic representations. Figure 4 shows that the optimum value for retained harmonics is 17 for the data.

The noise estimate according to equation (18) is 19.9 dB for straight line detrended error for the simulated data, to which 20 dB SNR pseudo random noise was added. Figure 5 represents an undersampled aircraft transient voltage measurement and Fig. 6, an aircraft current measurement. The nature of the contaminating noise is not known in these data. The estimate SNR for the voltage measurements of Fig. 5 is 18.9 dB.

The filtered versions of the data sets are given in Figs. 7-10. Aircraft current measurements plots are not overlaid as in the other data plots because of the difficulties in distinguishing distinct graphs. Figure 9 is merely an FFT reproduction of Fig. 6; Figure 10 is the filtered aircraft current data. Figures 7 and 8 show the performance of the filter in more comparable fashions and correspond to the data of Figs. 2 and 5, respectively. Figure 7 indicates the smoothing effects of the filter and its inability to remove all noise. Figure 8 shows that the filter "reconstructs"

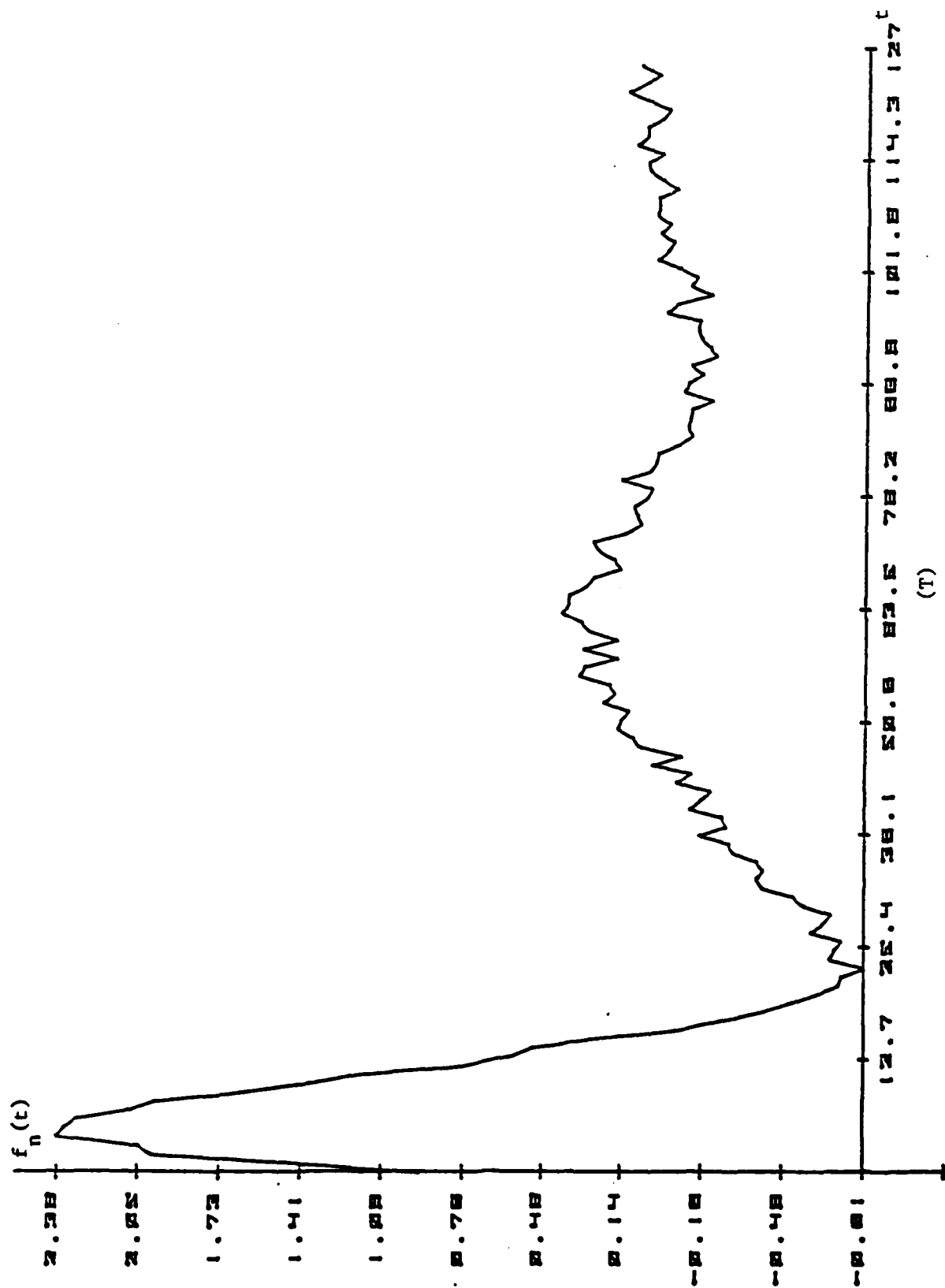


Figure 2: Simulated Noisy Data
($T = 20$ ms)

VARIATION OF ERROR FIGURE AT 20 dB

$$\omega_0 = 2.4933 \text{ rad/s}$$

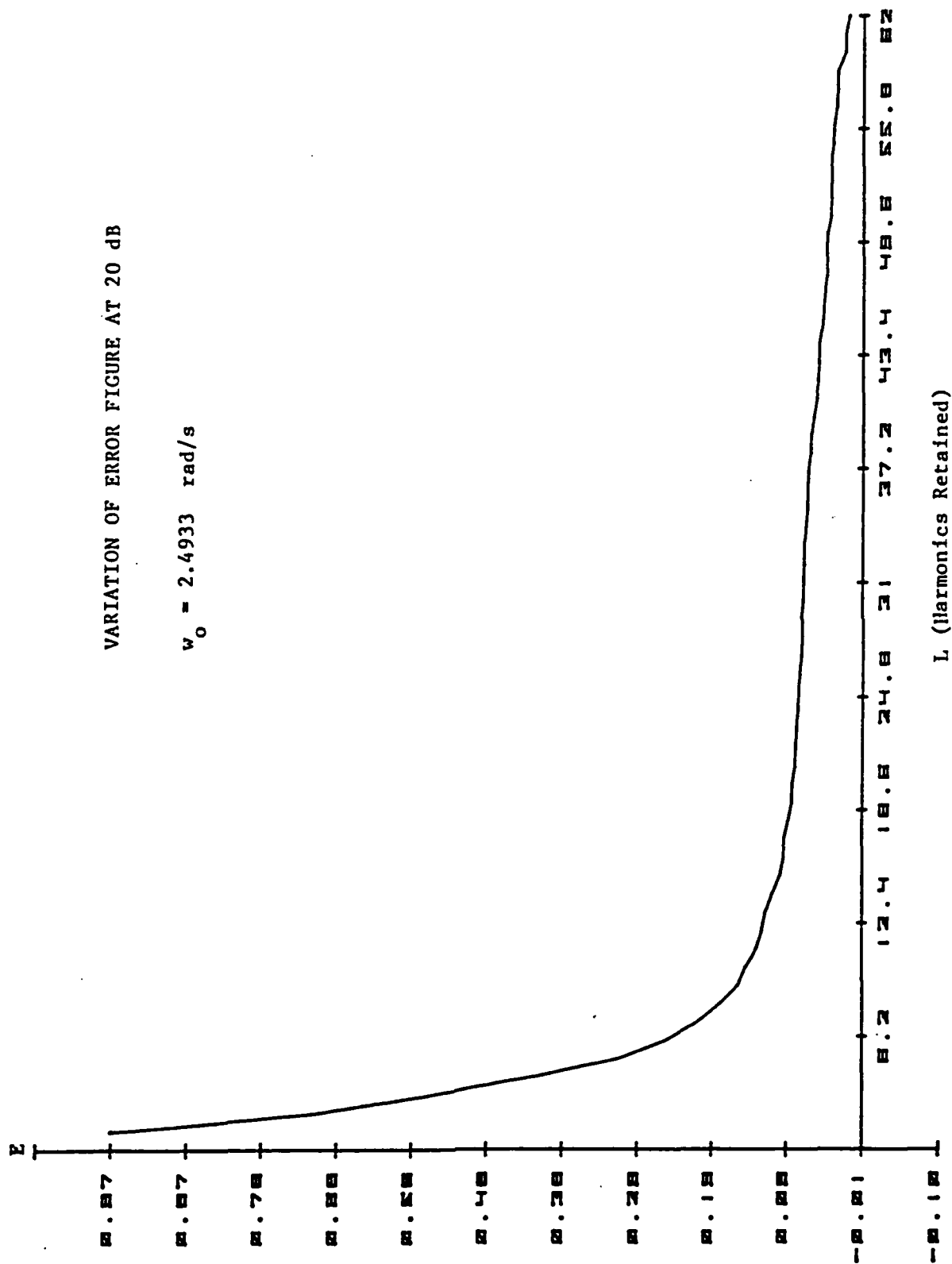


Figure 3: RMS Value of Error Signal

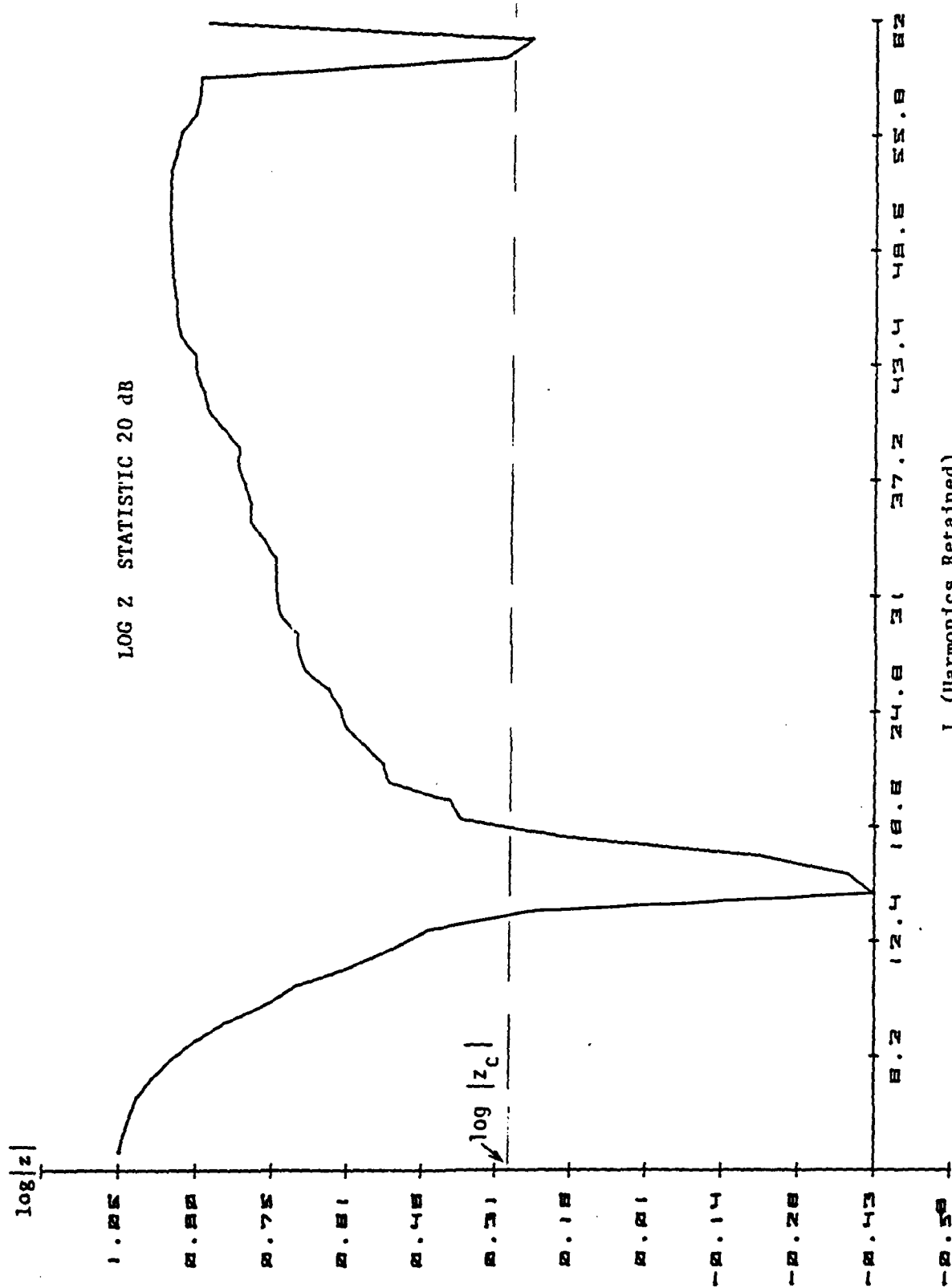


Figure 4: Variations of the Z-Statistic's Magnitude

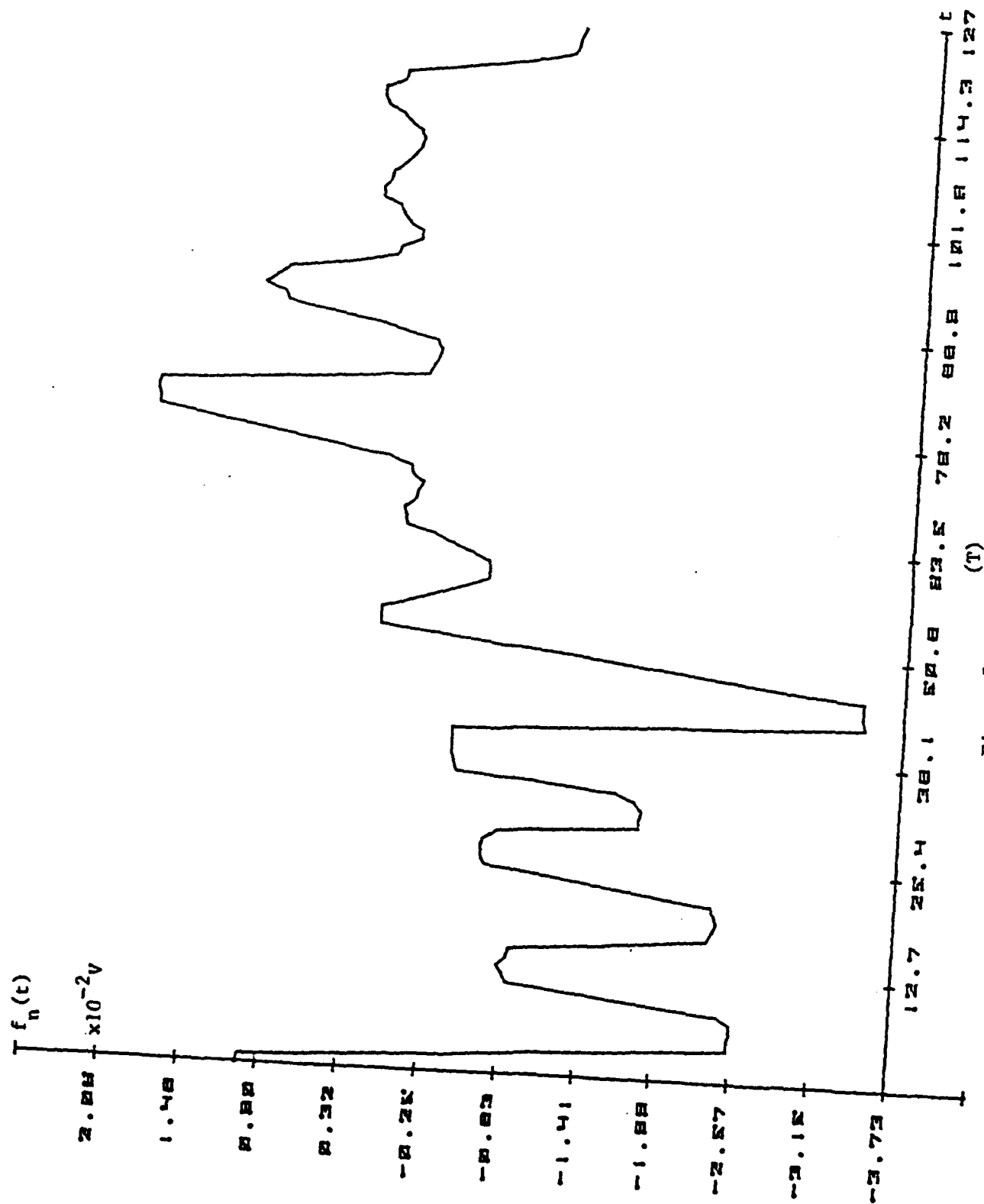
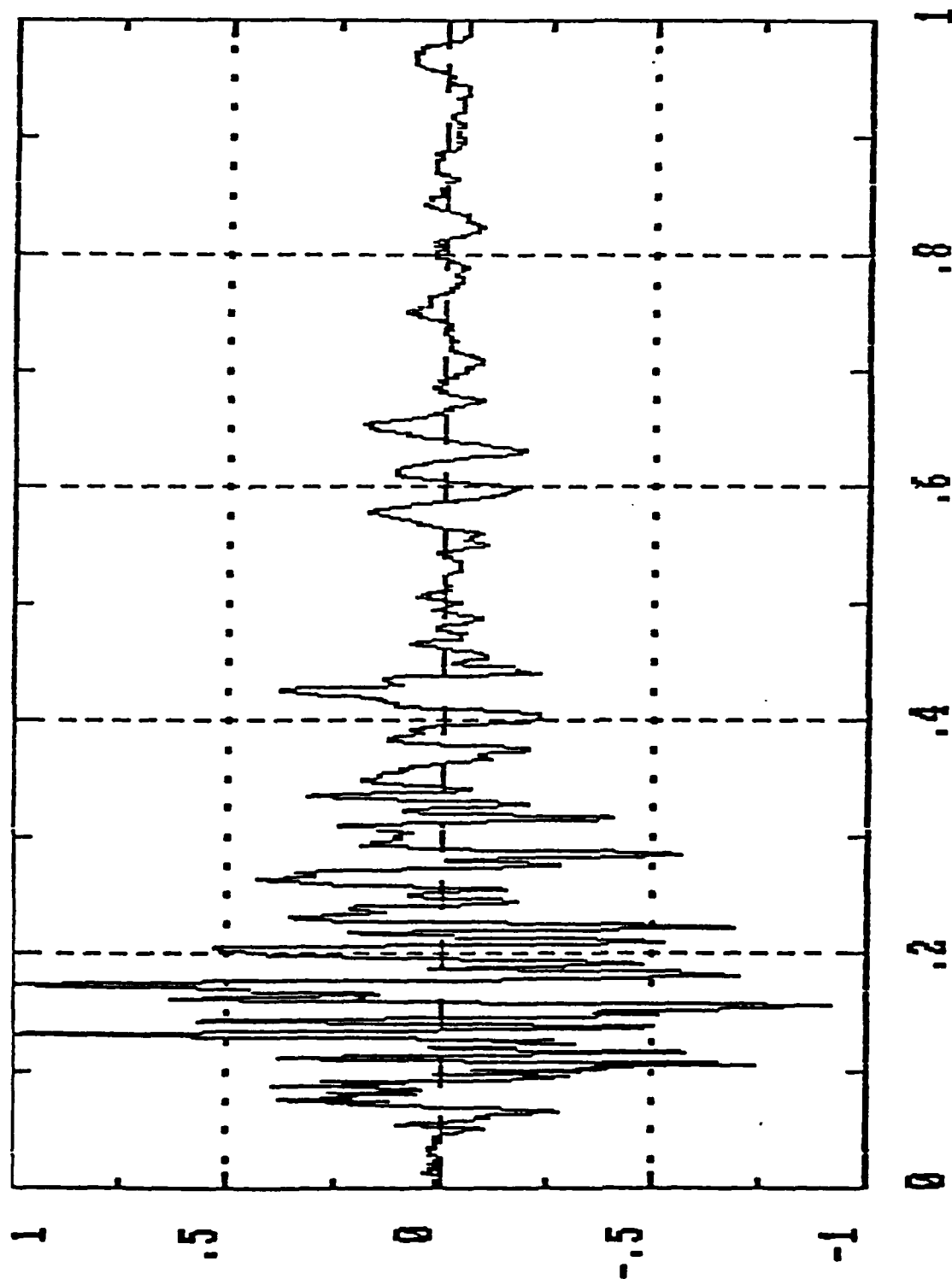


Figure 5: Aircraft Voltage Measurements
($T = 0.39063 \text{ ns}$)



PEAK AMPLITUDE= 5.96472E-01

TIME (full scale = 3.1E-06)

DATA SET E8527B002 BEFORE FILTERING

Figure 6: Aircraft Surface Current Measurements

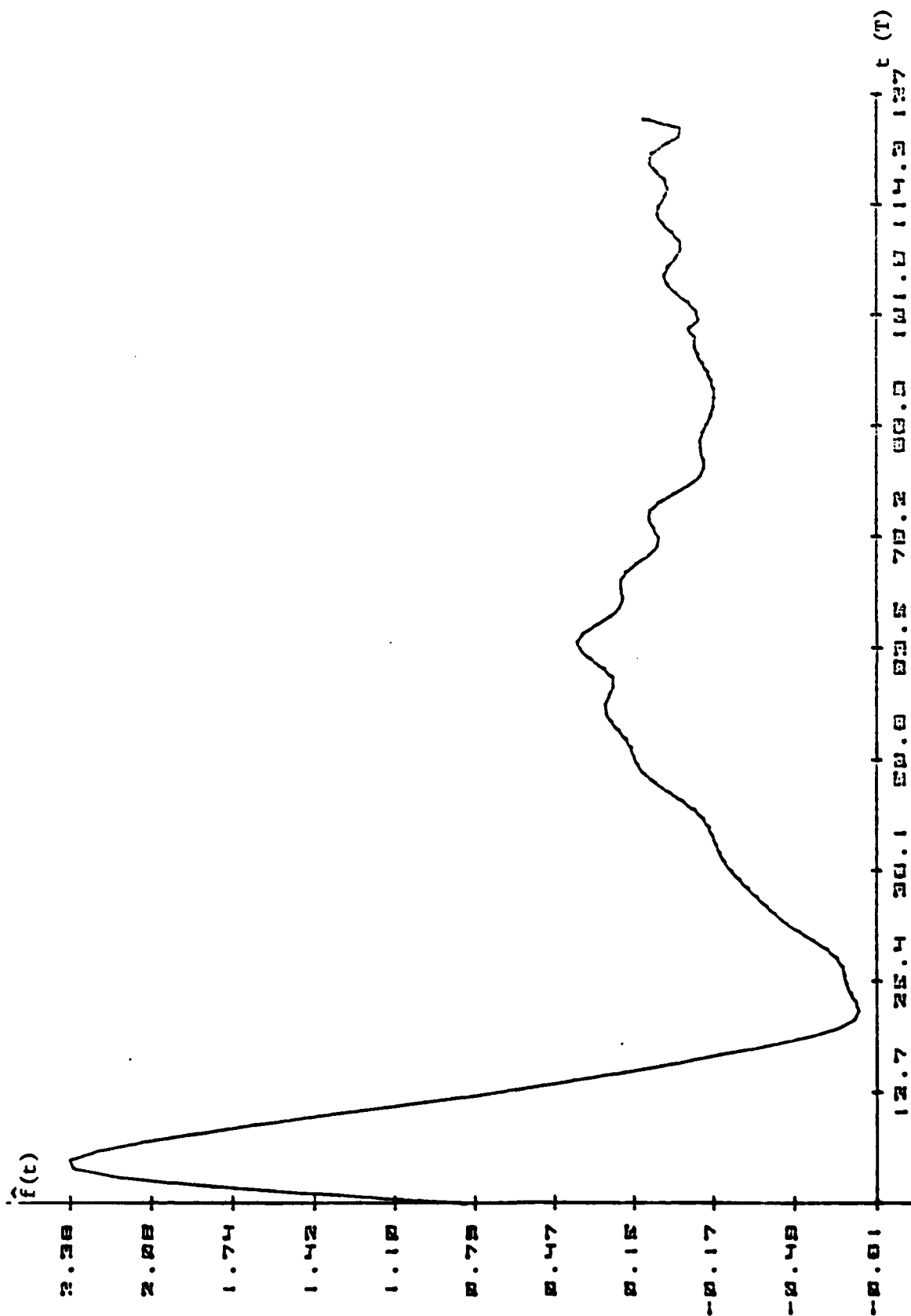


Figure 7: Filtered Simulated Data
($T=20ms$)

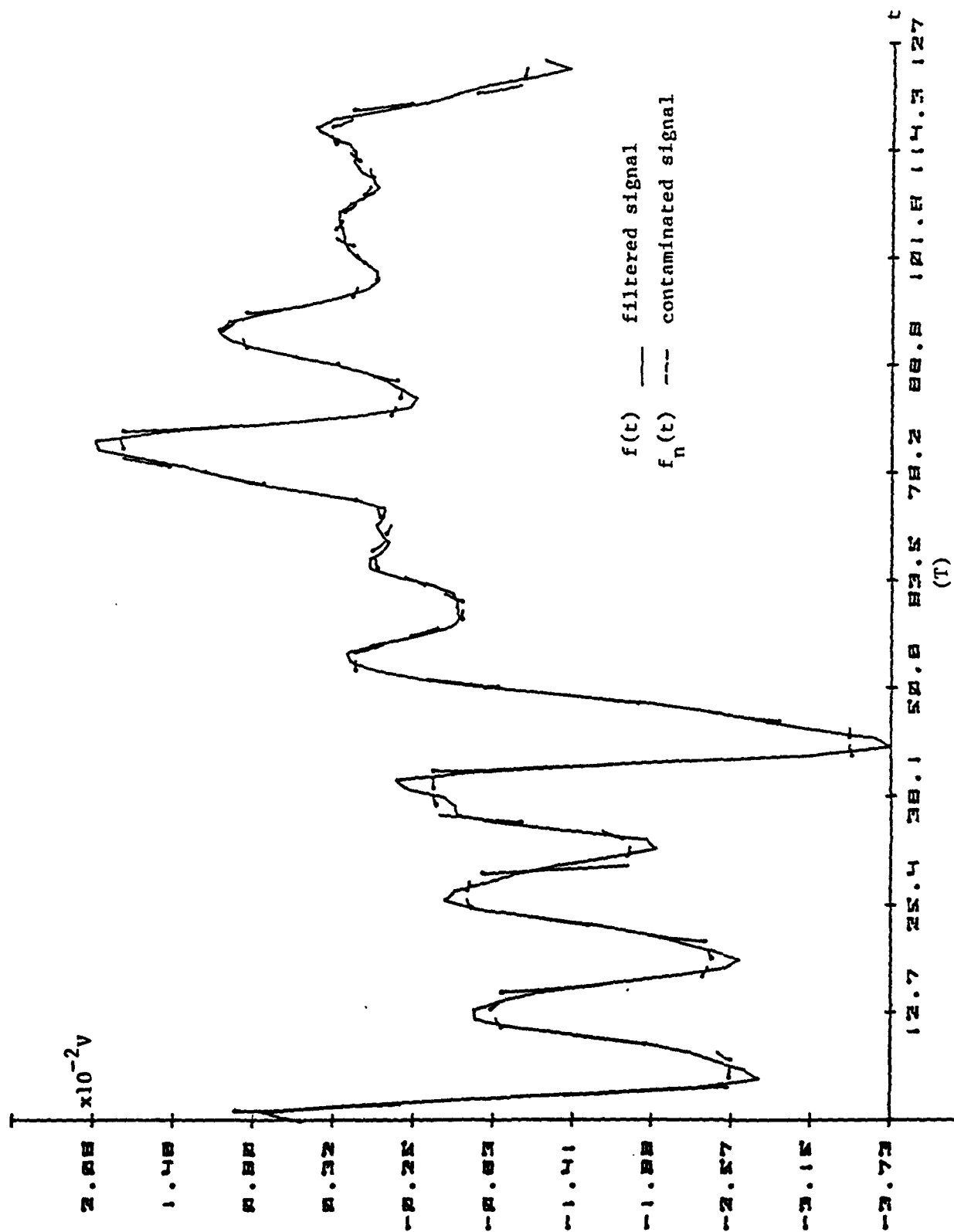
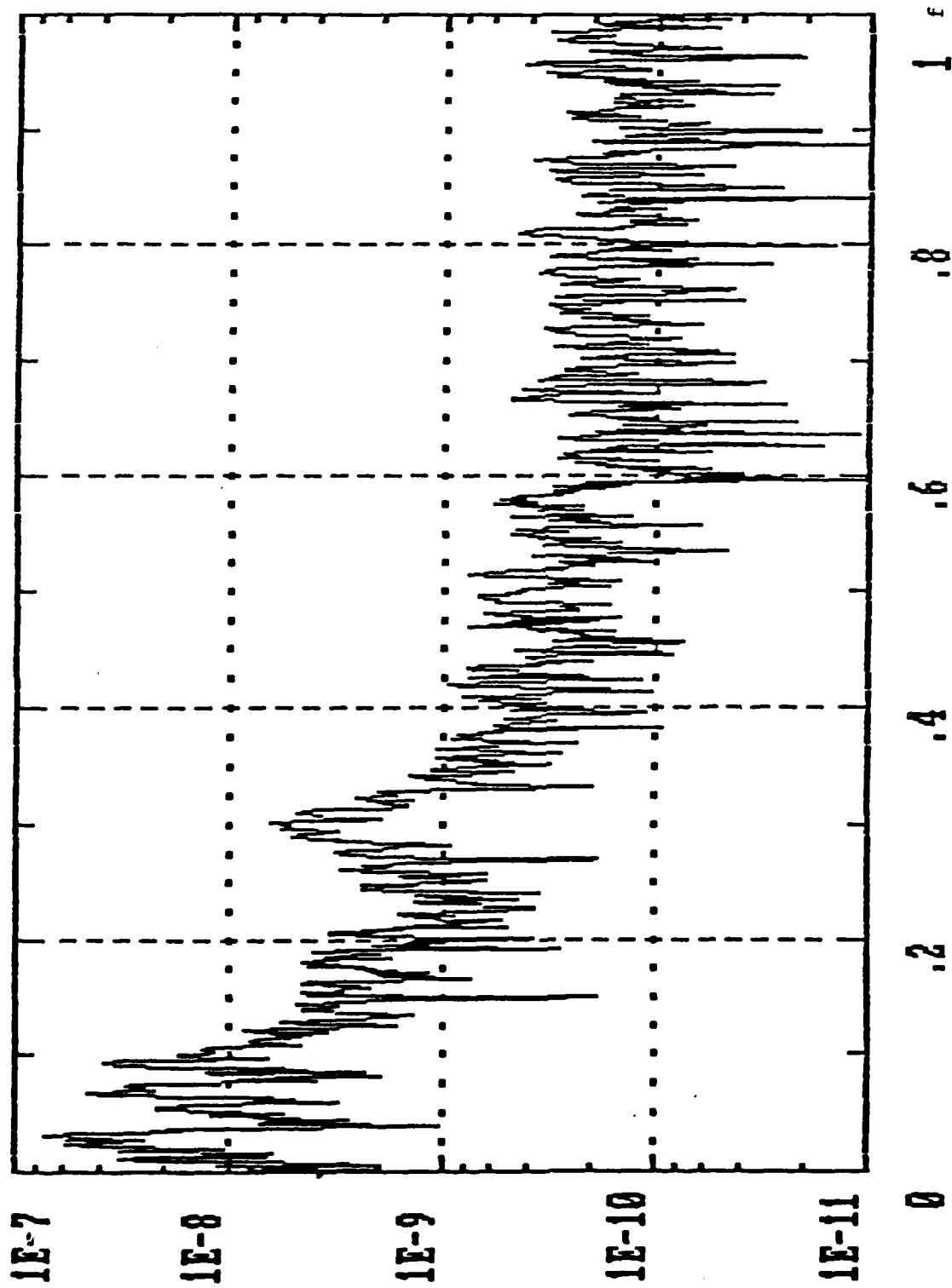


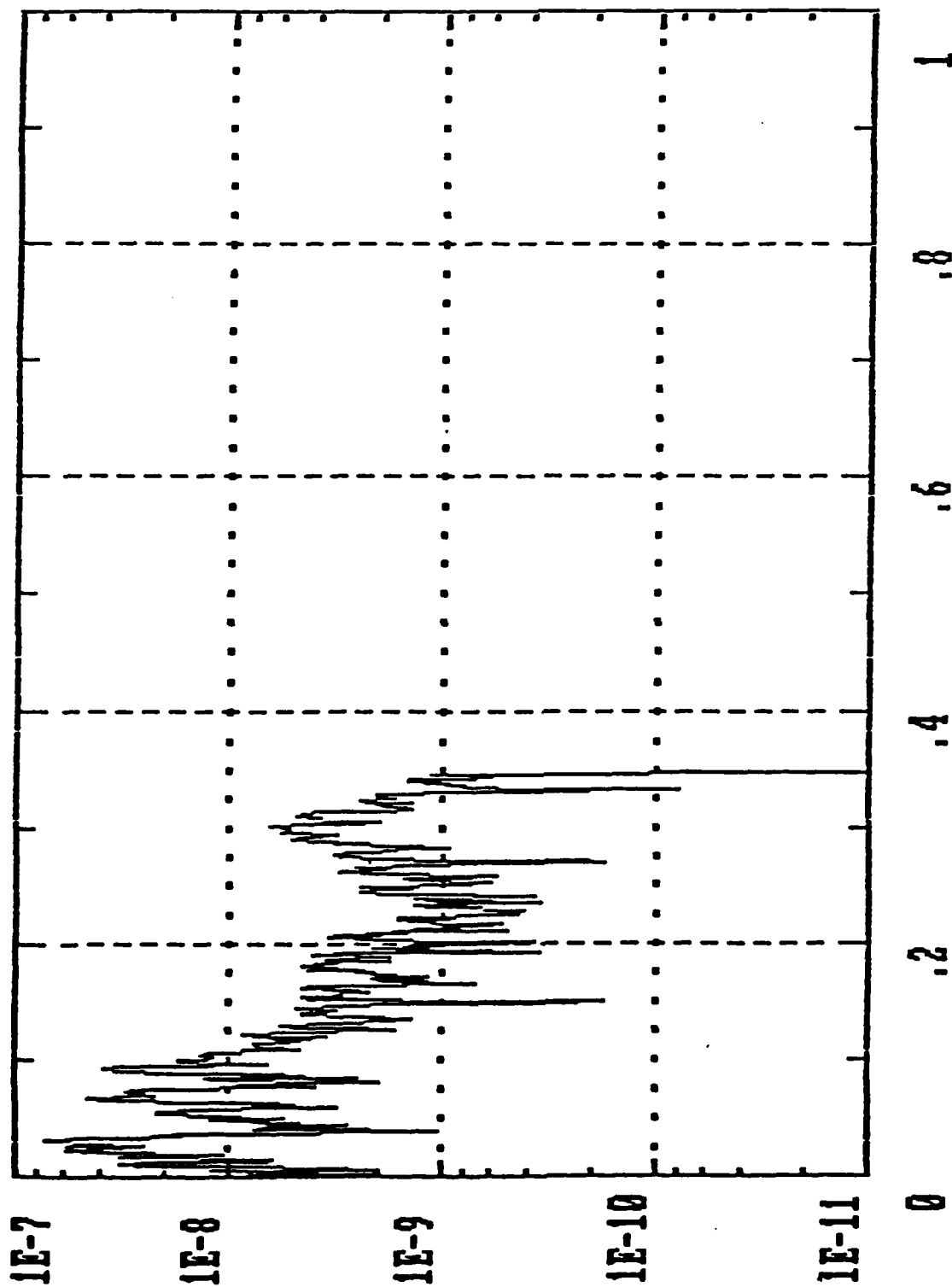
Figure 8: Aircraft Voltage Measurements
($T = 0.39063 \text{ ns}$)



PEAK AMPLITUDE= 7.02447E-08 FREQUENCY (full scale = 3.3E+08)

FFT OF DATA SET E8527B002 BEFORE FILTERING

Figure 9: FFT of Aircraft Current Measurement



PEAK AMPLITUDE= 7.03140E-08 FREQUENCY (full scale = 3.3E+08)

FFT OF DATA SET E8527B002 AFTER FILTERING

Figure 10: FFT of Filtered Aircraft Current Measurement

an intelligence estimate and, only in that sense, compensates for undersampling. Finally, Fourier coefficients for the data of Figs. 2 and 5 are respectively given in Tables 1 and 2.

For the data that are presented, linear interpolation is used to compute the Fourier coefficients. If the discrete Fourier series coefficients are used, the truncation of the series represents the result of applying an ideal zero-phase lowpass filter. With linear, quadratic, etc. interpolation used in equations (6) - (8), there will be some resulting phase and amplitude distortion, albeit very small. Each computational scheme seems to have its own advantages and disadvantages. However, the associated differences in the filtered signal are generally within the residual noise.

Conclusions

A simple technique for reducing random noise components in serial data has been presented. Basic theory required for determining the estimate signal has been given along with information on the curve-fit error single lag, autocorrelation coefficient. Statistically determined cutoff in harmonic content of signals has been used in truncating (Fourier) least squares curve-fits. Verification of the procedure was provided by considering examples of results when the filter is applied to simulated data sets as well as actual measured data where the noise characteristics are not known. These, together with the actual estimates of noise content, indicate that serial correlation filtering is a viable procedure for smoothing random data.

TABLE 1. FOURIER COEFFICIENTS OF SIMULATED DATA
(T_0 = Fundamental Period)

A0	W0	T_0
.119067+000	.249333+001	200000-001
HARMONIC	A	B
1	.240617+000	-.336733-001
2	.554411+000	.124682+000
3	.215013+000	.319436+000
4	.804515-001	.332959+000
5	-.317570-001	.257701+000
6	-.623676-001	.158842+000
7	-.568762-001	.105023+000
8	-.384506-001	.889833-001
9	-.460509-001	.560041-001
10	-.225412-001	.634750-001
11	-.299089-001	.469672-001
12	-.270673-001	.400109-001
13	-.187330-001	.247502-001
14	-.111443-001	.120933-001
15	-.225465-001	.109459-001
16	.244487-002	.304705-001
17	-.126514-001	.125419-001

TABLE 2. FOURIER COEFFICIENTS OF AIRCRAFT VOLTAGE MEASUREMENTS
(T_0 = Fundamental Period)

A0	W0	T_0
-.504385-002	.126652+009	.390630-009
HARMONIC	A	B
1	-.339958-002	-.116889-001
2	-.299620-003	.102555-002
3	-.982244-003	-.354796-002
4	.123119-002	-.149075-002
5	.301493-003	.211356-002
6	-.279008-002	-.310397-002
7	.232368-002	.155552-002
8	.177139-003	-.519210-003
9	.598966-003	-.419900-002
10	.491952-002	.410862-002
11	.155511-002	.186848-002
12	.332603-003	.425356-003
13	.148213-002	.100724-002
14	.728591-003	.892899-003
15	.324246-003	.503268-003
16	.125321-002	.555634-003
17	.870975-004	.378663-003
18	.422560-003	.300633-003
19	.120060-002	.810492-003
20	-.246067-003	.568349-003
21	.138480-003	.162885-003
22	.801859-003	.678513-003
23	-.341879-003	.615869-003
24	.129803-003	.288792-003
25	.799167-003	.208817-003

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APPENDICES

Appendix A

Let 1) k be the discrete variable where k is an element of $\{0, 1, \dots, N-1\}$ for an integer N ; 2) e_k and b_k be stationary random noises with arbitrary moments; 3) a_k and f_k be deterministic signals; 4) \hat{f}_k be the sum of a_k and $-b_k$; and 5) $\hat{f}_k = f_k - e_k$ for the signal f_k . It follows that $a_l = f_l$.

Proof: From conditions 5) and 4)

$$e_k = f_k - a_k + b_k \quad k=0, \dots, N-1. \quad (A-1)$$

Then the product $e_l e_k$ is given by

$$e_l e_k = f_l f_k - f_l a_k + f_l b_k - a_l f_k + a_l a_k - a_l b_k + b_l f_k - a_k b_l + b_l b_k. \quad (A-2)$$

where l and k are arbitrary elements of $\{0, 1, \dots, N-1\}$. The expected value of the product is

$$e[e_l e_k] = c \delta_k^l = (f_l - a_l)(f_k - a_k) + [(f_l - a_l) + (f_k - a_k)] e[b_l] \quad (A-3)$$

where condition 2) has been used, c is a constant and δ_k^l is the Kronecker delta.

In other words, if $(f_l - a_l) + (f_k - a_k)$ is not zero and $l \neq k$,

$$e[b_l] = \frac{-(f_l - a_l)(f_k - a_k)}{(f_l - a_l) + (f_k - a_k)} \quad (A-4)$$

$$(f_l - a_l) + (f_k - a_k) = 0. \quad (A-5)$$

From equation A-3 for $l \neq k$, since l is not always equal to k ,

$$f_l = a_l \quad l = 0, \dots, N-1. // \quad (A-5)$$

Thus, this proof shows that the deterministic part of the estimate \hat{f}_k is unique under the stated conditions 1) through 5).

Appendix B

Theorem 1

Let N and r be positive integers and a_i ($i=1, \dots, N$) be real numbers. If μ is a bounded non-negative real number such that $|a_i| \leq \mu$ ($i=1, \dots, N$), then there is a sequence of real numbers $\{A_j\}$ ($j=1, \dots, r$) for which

$$\frac{1}{N} \left| \sum_{i=1}^N a_i^r \right| \leq A_r \quad r=1, 2, \dots$$

for all N .

Lemma If C and D are non-negative real numbers such that $C \leq D$, then $C^r \leq D^r$ ($r=1, 2, \dots$).

Proof of Lemma: For $r=1$ $C \leq D$, by hypothesis. For m an integer less than or equal to r , if $C^{m-1} \leq D^{m-1}$ then $DC^{m-1} \leq D^m$. However, the inductive process allows the case $C^{m-(m-1)} \leq D$ which implies $C^m \leq DC^{m-1}$. Thus, $C^m \leq D^m$. //

Proof of Theorem 1: Suppose $A_r = \mu^r$. By hypothesis, $\mu^j \geq 0$ ($j=1, \dots, r$). The hypothesis and the above lemma imply that

$$\sum_{i=1}^N |a_i|^r \leq N\mu^r.$$

The triangle of inequality rule and the rule of exponentiation imply that

$$\frac{1}{N} \left| \sum_{i=1}^N a_i^r \right| \leq \mu^r \quad N, r=1, 2, \dots //$$

Theorem 2

Let N be a positive integer, a_i be real numbers ($i=1, \dots, N$) and \bar{a} be equal to $\frac{1}{N} \sum_{i=1}^N a_i$. If $a_i \neq \bar{a}$ ($i=1, \dots, N$) and β is a real number equal to $\min_i [a_i - \bar{a}]^2$, then there is a real number γ ($\gamma > 0$) such that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (a_i - \bar{a})^2 \geq \gamma.$$

Proof of Theorem 2: Suppose $\gamma = \beta$. By hypothesis, $\beta > 0$. Also,

$$\frac{1}{N} \sum_{i=1}^N (a_i - \bar{a})^2 \geq \frac{1}{N} N\beta = \beta$$

therefore,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N (a_i - \bar{a})^2 \geq \beta. //$$

END

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